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Homoclinic orbits in a first order superquadratic Hamiltonian system

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0. Introduction

In this article we consider the following first order Hamiltonian system:

$$\dot{z}(t) = JH_z(t, z(t)), \tag{HS}$$

where $\bullet = \frac{d}{dt}$, $z = (z_1, \cdots, z_{2N}) \in \mathbf{R}^{2N}$,

$$J = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

and $H(t,z) \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$. We denote by $(\,\cdot\,,\,\cdot\,)$ the standard inner product in $\mathbf{R}^{2N}$ and throughout this article, we assume $H(t,z)$ has the following form:

$$H(t,z) = \frac{1}{2}(Az, z) + W(t,z), \tag{0.1}$$

where

(A) A is a $2N \times 2N$ symmetric matrix such that

$$\sigma(JA) \cap i\,\mathbf{R} = \emptyset$$

and $W(t,z)$ is a 2π-periodic and *globally superquadratic* function, more precisely $W(t,z)$ satisfies

(W1) $W(t,z) \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is 2π-periodic in t and $W(t,0) \equiv 0$,

(W2) there is an $\mu > 2$ such that

$$\mu W(t,z) \leq (W_z(t,z), z) \quad \text{for all } (t,z) \in \mathbf{R} \times \mathbf{R}^{2N},$$

(W3) there are $\alpha \geq \mu$ and $k_1 > 0$ such that

$$k_1 \mid z \mid^{\alpha} \leq W(t,z) \quad \text{for all } (t,z) \in \mathbf{R} \times \mathbf{R}^{2N},$$

(W4) there are k_2, $k_3 > 1$ such that

$$| W_z(t,z) | \le k_2(W_z(t,z),z) + k_3 \quad \text{for all } (t,z) \in \mathbf{R} \times \mathbf{R}^{2N},$$

(W5) $W_z(t,z) = o(| z |)$ at $z = 0$ uniformly in $t \in \mathbf{R}$.

Under the above conditions, we study the existence of (nontrivial) homoclinic orbits emanating from 0. In other words, we consider the existence of solutions of (HS) such that

$$z(t) \to 0 \quad \text{as } | t | \to \infty. \tag{0.2}$$

We remark that 0 is an equilibrium point of (HS).

The existence of homoclinic orbits is studied by Coti-Zelati, Ekeland and Sere [**2**] and Hofer and Wysocki [**6**]. More precisely, under the conditions of (A), (W1), (W2), (W3) with $\alpha = \mu$, and

(W4') there is a $k_2 > 0$ such that

$$| W_z(t,z) | \le k_2 | z |^{\mu-1} \quad \text{for all } (t,z) \in \mathbf{R} \times \mathbf{R}^{2N},$$

and strict convexity of $W(t,z)$ with respect to z, [**2**] used a *dual variational formulation* and obtained the existence of homoclinic orbits. On the other hand, [**6**] studied (HS) under conditions (A), (W1), (W2), (W3) with $\alpha = \mu$, and (W4'). They used first order elliptic system and nonlinear Fredholm operator theory and obtained the existence of a homoclinic orbit. See also [**1,5,9,10,11,12**] for similar problems for second order Hamiltonian systems. We remark that (W4) is a weaker condition than (W4') under the conditions (W2) and (W3).

We take another approach to this problem. We study the convergence of *subharmonic* solutions to a nontrivial homoclinic solution; that is, we consider $2\pi T$-periodic solutions $z_T(t)$ ($T \in \mathbf{N}$) of (HS), which possess some minimax characterization, and try to pass to the limit as $T \to \infty$.

In case where A satisfies

(A^c) A is a $2N \times 2N$ symmetric matrix such that

$$\sigma(JA) \cap i\mathbf{R} \neq \emptyset,$$

the behavior of $(z_T(t))_{T \in \mathbf{N}}$ as $T \to \infty$ is studied by Rabinowitz [**7**] and Felmer [**4**]. They showed

$$\|z_T(t)\|_{L^\infty} \to 0 \quad \text{as } T \to \infty \tag{0.3}$$

under suitable conditions on $W(t,z)$ and eigenvalues of JA.

Under the assumption (A), we remark that $0 \in \mathbf{R}^{2N}$ is a *hyperbolic point* of (HS) and (0.3) cannot take place in our setting of problem. Our main result is the following theorem, which is in contrast to the result of [**4**, **7**] and also ensures *the existence of a homoclinic orbit of* (HS).

Theorem 0.1([13]). *Assume (A) and (W1)-(W5). Then there is a sequence* $(z_T(t))_{T\in\mathbf{N}} \subset C^1(\mathbf{R}, \mathbf{R}^{2N})$ *of solutions of* (HS) *such that*

(i) $z_T(t)$ *is a* $2\pi T$*-periodic solution of* (HS)*;*

(ii) *there are constants* m, $M > 0$ *independent of* $T \in \mathbf{N}$ *such that*

$$m \le \int_0^{2\pi T} [\frac{1}{2}(-J\,\dot{z}_T, z_T) - H(t, z_T)]\,dt \le M; \tag{0.4}$$

(iii) *moreover* $(z_T(t))_{T\in\mathbf{N}}$ *is compact in the following sense; for any sequence of integers* $T_n \to \infty$, *there is a subsequence* (T_{n_k}) *and a (nontrivial) homoclinic orbit* $z_\infty(t)$ *emanating from* 0 *such that*

$$z_{T_{n_k}}(t) \to z_\infty(t) \quad \textit{in } C^1_{loc}(\mathbf{R}, \mathbf{R}^{2N}).$$

Remark 0.1. In case where $W(t,z)$ does not depend on $t \in \mathbf{R}$, the conclusion of the above theorem holds without assumption (W4). That is,

Theorem 0.2 ([13]). *Assume (A), (W1)-(W3), (W5) and* $W(z)$ *is independent of* $t \in \mathbf{R}$. *Then the conclusion of Theorem 0.1 holds.*

We also remark that the convergence of $2\pi T$-periodic solutions to a nontrivial homoclinic orbit is obtained for a second order Hamiltonian system by Rabinowitz **[10]** and our work is largely motivated by it.

In this note, we assume the following growth condition (W4") on $W(t,z)$

(W4") there are $\beta \in [\alpha, \alpha+1)$ and $k_2 > 0$ such that

$$|\, W_z(t,z) \,| \le k_2 \,|\, z \,|^{\beta-1} \quad \text{for all } (t,z) \in \mathbf{R}\times\mathbf{R}^{2N}.$$

instead of (W4) and (W5) (clearly (W4) and (W5) follow from (W4") under the condition (W2)) and we prove the following Theorem 0.3 rather than Theorems 0.1 and 0.2 for the sake of simplicity.

Theorem 0.3. *Assume (A), (W1)-(W3) and (W4"). Then the conclusion of Theorem 0.1 holds.*

For the proof of Theorems 0.1 and 0.2, we refer to **[13]**. In Section 1, we deal with $2\pi T$-periodic solutions of (HS); we introduce a variational formulation and minimax procedure and we prove the existence of $2\pi T$-periodic solutions $z_T(t)$ of (HS). At the same time, we obtain uniform estimates (from above and from below) of corresponding critical values. In Section 2, we get uniform estimates of $z_T(t)$ and pass to the limit as $T \to \infty$ and complete the proof of Theorem 0.3. Finally in Section 3, we give a proof to Proposition 1.1; we study properties of the operator $J\frac{d}{dt}+A$, especially, the L^p-boundedness of some projection operators related to $J\frac{d}{dt}+A$. These properties are used in Sections 1 and 2 without proof.

1. $2\pi T$-periodic solutions of (HS)

In this section we study the following problem:

$$\begin{aligned} \dot{z} &= JH_z(t,z), && \text{in } \mathbf{R} \\ z(t+2\pi T) &= z(t), && \text{in } \mathbf{R} \end{aligned} \tag{HS:T}$$

where $T \in \mathbf{N}$.

There is a one-to-one correspondence between solutions of (HS:T) and critical points of the functional:

$$\begin{aligned} I_T(z) &= -\frac{1}{2}\int_0^{2\pi T}(J\dot{z}, z)dt - \int_0^{2\pi T} H(t, z(t))dt \\ &= \frac{1}{2}\int_0^{2\pi T}(-J\dot{z} - Az, z)dt - \int_0^{2\pi T} W(t, z(t))dt. \end{aligned} \tag{1.1}$$

So we will seek for a nontrivial critical points of $I_T(z)$.

In what follows, for $p \in [1, \infty)$ we denote by $L^p_{2\pi T}$ the space of $2\pi T$-periodic functions $\mathbf{R} \to \mathbf{R}^{2N}$ whose p-th powers are integrable on $(0, 2\pi T)$. We use the notations

$$\|z\|_{L^p_{2\pi T}} = \left(\int_0^{2\pi T} | z(t) |^p \, dt\right)^{1/p}, \tag{1.2}$$

and

$$(z, w)_{2\pi T} = \int_0^{2\pi T} (z(t), w(t))dt \tag{1.3}$$

for $z \in L^p_{2\pi T}$ and $w \in L^q_{2\pi T}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\Phi_{2\pi T} = -(J\frac{d}{dt} + A) : D(\Phi_{2\pi T}) \subset L^2_{2\pi T} \to L^2_{2\pi T}$ be a self-adjoint operator under periodic boundary conditions. In Section 3 we will see

$$(-a, a) \cap \sigma(\Phi_{2\pi T}) = \emptyset \quad \text{for some } a > 0. \tag{1.4}$$

We consider the absolute value $| \Phi_{2\pi T} |$ of $\Phi_{2\pi T}$ and let

$$E_{2\pi T} = D(| \Phi_{2\pi T} |^{1/2})$$

and

$$\|z\|_{E_{2\pi T}} = \| \mid \Phi_{2\pi T} \mid^{1/2} z\|_{L^2_{2\pi T}} \quad \text{for } z \in E_{2\pi T}.$$

By (1.4), $E_{2\pi T}$ has an orthogonal decomposition:

$$E_{2\pi T} = E^+_{2\pi T} \oplus E^-_{2\pi T} \tag{1.5}$$

where the quadratic form: $z \mapsto (\Phi_{2\pi T} z, z)_{2\pi T}$ is positive (resp. negative) definite on $E^+_{2\pi T}$ (resp. $E^-_{2\pi T}$). We denote by

$$P^{\pm}_{2\pi T} : E_{2\pi T} \to E^{\pm}_{2\pi T} \tag{1.6}$$

the corresponding orthogonal projections. Then we have

$$(\Phi_{2\pi T} z, z)_{2\pi T} = \|P^+_{2\pi T} z\|^2_{E_{2\pi T}} - \|P^-_{2\pi T} z\|^2_{E_{2\pi T}} \quad \text{for all } z \in E_{2\pi T}. \tag{1.7}$$

We can see

$$\begin{aligned} I_T(z) &= \frac{1}{2}(\Phi_{2\pi T} z, z)_{2\pi T} - \int_0^{2\pi T} W(t,z)dt \\ &= \frac{1}{2}\|P^+_{2\pi T} z\|^2_{E_{2\pi T}} - \frac{1}{2}\|P^-_{2\pi T} z\|^2_{E_{2\pi T}} - \int_0^{2\pi T} W(t,z)dt \end{aligned}$$

The following properties of $E_{2\pi T}$ and $P^{\pm}_{2\pi T}$ will be proved in Section 3.

Proposition 1.1.

(i) *Let $H^{1/2}_{2\pi T}$ be a completion of* $\text{span}\{ae^{ijt/T} + \bar{a}e^{-ijt/T};\ j \in \mathbf{N},\ a \in \mathbf{C}^{2N}\}$ *under the norm*

$$\|z\|^2_{H^{1/2}_{2\pi T}} = 2\pi T \sum_{j \in \mathbf{Z}} (1 + \frac{\mid j \mid}{T}) \mid a_j \mid^2$$

where

$$z(t) = \sum_{j \in \mathbf{Z}} a_j e^{ijt/T} \quad (a_j \in \mathbf{C}^{2N},\ a_{-j} = \bar{a}_j).$$

Then $E_{2\pi T} = H^{1/2}_{2\pi T}$ and there are constants c_0, $c_0' > 0$ independent of $T \in \mathbf{N}$ such that

$$c_0\|z\|_{H^{1/2}_{2\pi T}} \le \|z\|_{E_{2\pi T}} \le c_0'\|z\|_{H^{1/2}_{2\pi T}} \tag{1.8}$$

for all $z \in E_{2\pi T}$.

(ii) *For any $p \in [2, \infty)$, there is a constant $c_p > 0$ independent of $T \in \mathbf{N}$ such that*

$$\|z\|_{L^p_{2\pi T}} \le c_p\|z\|_{E_{2\pi T}} \quad \text{for all } z \in E_{2\pi T}. \tag{1.9}$$

Moreover, the embedding $E_{2\pi T} \to L^p_{2\pi T}$ is compact for all $T \in \mathbf{N}$ and $p \in [2, \infty)$.

(iii) *There is a constant $c > 0$ independent of $T \in \mathbf{N}$ such that*

$$\|z\|_{L^\infty} \le c\|\Phi_{2\pi T} z\|_{L^2_{2\pi T}} \quad \text{for all } z \in D(|\Phi_{2\pi T}|). \tag{1.10}$$

(iv) *For any $p \in (1, \infty)$, there is a constant $\bar{c}_p > 0$ independent of $T \in \mathbf{N}$ such that*

$$\|P^\pm_{2\pi T} z\|_{L^p_{2\pi T}} \le \bar{c}_p \|z\|_{L^p_{2\pi T}} \quad \text{for all } z \in E_{2\pi T}. \tag{1.11}$$

By (1.7) and (ii) of Proposition 1.1, we have $I_T(z) \in C^1(E_{2\pi T}, \mathbf{R})$. Moreover we have the Palais-Smale compactness condition. This condition is required when we apply minimax methods to $I_T(z)$.

Proposition 1.2. *$I_T(z)$ satisfies the following Palais-Smale compactness condition:*

(P.S.) *Whenever a sequence $(z_j)_{j=1}^\infty$ in $E_{2\pi T}$ satisfies for some $M > 0$,*

$$|I_T(z_j)| \le M \qquad \text{for all } j,$$

$$I_T'(z_j) \to 0 \quad \text{in } E^*_{2\pi T} \qquad \text{as } j \to \infty,$$

there is a subsequence of $(z_j)_{j=1}^\infty$ which converges in $E_{2\pi T}$.

Proof. As in [**8**, Chapter 6]. ∎

To find a nontrivial critical point of $I_T(z)$, we use the following proposition which is a special case of a theorem of Rabinowitz [**8**, Theorem 5.29].

In what follows, $B_r(E)$ denotes the open ball of radius r in a Hilbert space E and $\partial B_r(E)$ denotes its boundary.

Proposition 1.3. *Let E be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. Suppose E admits an orthogonal decomposition $E = E^+ \oplus E^-$ and $I(u) \in C^1(E, \mathbf{R})$ satisfies the Palais-Smale compactness condition and the following conditions:*

1° *$I(u) = \frac{1}{2}\langle P^+u - P^-u, u\rangle + b(u)$, where $P^\pm : E \to E^\pm$ are the orthogonal projectors, $b'(u)$ is compact,*

2° *there are constants $m, \rho > 0$ such that $I|_{\partial B_\rho(E^+)} \ge m$, and*

3° *there is an $e \in \partial B_1(E^+)$ and $R > \rho$ such that*

$$I|_{\partial N} \le 0$$

where $N = \{u + re; u \in B_R(E^-),\ 0 < r < R\}$.

Then $I(u)$ possesses a critical value $b \ge m$ which can be characterized as

$$b = \inf_{h \in \Gamma} \sup_{u \in N} I(h(1, u)) \ge m,$$

where

$$\Gamma = \{h \in C([0,1] \times E, E); h \, satisfies \, (\Gamma_1)\text{–}(\Gamma_3)\}.$$

Here

(Γ_1) $h(0,u) = u$ *for all* $u \in N$,

(Γ_2) $h(t,u) = u$ *for* $u \in \partial N$ *and* $t \in [0,1]$, *and*

(Γ_3) $h(t,u) = e^{\theta(t,u)(P^+-P^-)}u + K(t,u)$, *where* $\theta \in C([0,1] \times E, \mathbf{R})$ *and* K *is compact.* ∎

We will apply the above proposition to $I = I_T$, $E^\pm = E^\pm_{2\pi T}$ and $e = e_T \equiv P^+_{2\pi T}\varphi$, where $\varphi \in C_0^\infty((0,2\pi), \mathbf{R}^{2N})$ is a function such that

$$\int_0^{2\pi} ((-J\frac{d}{dt} - A)\varphi, \varphi)dt > 0.$$

(We extend φ to $(0, 2\pi T) \to \mathbf{R}^{2N}$ by setting $\varphi = 0$ on $[2\pi, 2\pi T)$ and we regard it as a $2\pi T$-periodic function on $\mathbf{R}$.)

Lemma 1.4. (i) *There are constants* a_1, $a_2 > 0$ *independent of* $T \in \mathbf{N}$ *such that*

$$a_1 \le \|e_T\|_{E_{2\pi T}} \le a_2 \quad \textit{for all } T \in \mathbf{N}. \tag{1.12}$$

(ii) *For any* $p \in (1, \infty)$, *there are constants* $a_{3,p}$, $a_{4,p} > 0$ *independent of* $T \in \mathbf{N}$ *such that*

$$a_{3,p} \le \|e_T\|_{L^p_{2\pi T}} \le a_{4,p} \quad \textit{for all } T \in \mathbf{N}. \tag{1.13}$$

Proof. (i) For any $T \in \mathbf{N}$, we have

$$\begin{aligned}
\|e_T\|^2_{E_{2\pi T}} &= \|P^+_{2\pi T}\varphi\|^2_{E_{2\pi T}} \ge \|P^+_{2\pi T}\varphi\|^2_{E_{2\pi T}} - \|P^-_{2\pi T}\varphi\|^2_{E_{2\pi T}} \\
&= (\Phi_{2\pi T}\varphi, \varphi)_{2\pi T} \\
&= \int_0^{2\pi} ((-J\frac{d}{dt} - A)\varphi, \varphi)dt \equiv a_1^2 > 0.
\end{aligned}$$

This shows the left hand side inequality of (1.12). Using (1.8), we have

$$\begin{aligned}
\|e_T\|^2_{E_{2\pi T}} &= \|P^+_{2\pi T}\varphi\|^2_{E_{2\pi T}} \le \|\varphi\|^2_{E_{2\pi T}} \\
&\le c_0'\|\varphi\|^2_{H^{1/2}_{2\pi T}} \le c_0'\|\varphi\|^2_{H^1_{2\pi T}} \\
&= c_0' \int_0^{2\pi} (|\, \varphi\,|^2 + |\, \dot{\varphi}\,|^2)dt \equiv a_2^2 < \infty.
\end{aligned}$$

Thus we get the right hand side inequality of (1.12).

(ii) By (1.12) and (1.9), we have the right hand side inequality of (1.13). To get the left hand side inequality, we observe for $\frac{1}{p}+\frac{1}{q}=1$

$$\|P_{2\pi T}^{+}\varphi\|_{L_{2\pi T}^{p}}\left(\int_{0}^{2\pi}|(J\frac{d}{dt}+A)\varphi|^{q}\,dt\right)^{1/q}\geq(P_{2\pi T}^{+}\varphi,-(J\frac{d}{dt}+A)\varphi)_{2\pi T}$$
$$\geq(\varphi,-(J\frac{d}{dt}+A)\varphi)_{2\pi T}=\int_{0}^{2\pi}((-J\frac{d}{dt}-A)\varphi,\varphi)dt.$$

Hence we have

$$\|e_T\|_{L_{2\pi T}^{p}}\geq\left(\int_{0}^{2\pi}((-J\frac{d}{dt}-A)\varphi,\varphi)dt\right)\left(\int_{0}^{2\pi}|(J\frac{d}{dt}+A)\varphi|^{q}\,dt\right)^{-1/q}\equiv a_{3,p}>0. \quad \blacksquare$$

We remark that $e_T\neq 0$ follows from Lemma 1.4.
Next we verify the assumptions 2° and 3° of Proposition 1.3.

Lemma 1.5. *There are constants ρ, $m>0$ independent of $T\in\mathbf{N}$ such that*

$$I_T(z)\geq m\quad\text{for all } z\in E_{2\pi T}^{+}\text{ with }\|z\|_{E_{2\pi T}}=\rho. \tag{1.14}$$

Proof. For $z\in E_{2\pi T}^{+}$, we have form (W4")

$$I_T(z)=\frac{1}{2}\|z\|_{E_{2\pi T}}^{2}-\int_{0}^{2\pi T}W(t,z)dt$$
$$\geq\frac{1}{2}\|z\|_{E_{2\pi T}}^{2}-k_2\|z\|_{L_{2\pi T}^{\beta}}^{\beta}.$$

By (1.9),

$$I_T(z)\geq\frac{1}{2}\|z\|_{E_{2\pi T}}^{2}-k_2c_\beta^{\beta}\|z\|_{E_{2\pi T}}^{\beta}.$$

Choosing $\rho=(\beta k_2c_\beta^{\beta})^{-1/(\beta-2)}$ and $m=(\frac{1}{2}-\frac{1}{\beta})(\beta k_2c_\beta^{\beta})^{-2/(\beta-2)}$, we get (1.14). $\blacksquare$

Lemma 1.6. *There is a constant $R>0$ which is independent of $T\in\mathbf{N}$ such that*

$$I\,|_{\partial N_{T,R}}\leq 0,$$

where

$$N_{T,R}=\{u+re_T\in E_{2\pi T};u\in B_R(E_{2\pi T}^{-}),\ 0<r<R\}.$$

Moreover there is a constant $M>0$ which is independent of $T\in\mathbf{N}$ such that

$$\sup_{z\in N_{T,R}}I_T(z)\leq M\quad\text{for all } T\in\mathbf{N}. \tag{1.15}$$

Proof. For $z = u + re_T$ $(u \in E^-_{2\pi T},\ r > 0)$, we have from (W3) that

$$\begin{aligned} I_T(u + re_T) &= \frac{r^2}{2}\|e_T\|^2_{E_{2\pi T}} - \frac{1}{2}\|u\|^2_{E_{2\pi T}} - \int_0^{2\pi T} W(t, u + re_T)dt \\ &\le \frac{r^2}{2}\|e_T\|^2_{E_{2\pi T}} - \frac{1}{2}\|u\|^2_{E_{2\pi T}} - k_1\|u + re_T\|^\alpha_{L^\alpha_{2\pi T}}. \end{aligned}$$

By (1.11), we have

$$r\|e_T\|_{L^\alpha_{2\pi T}} = \|P^+_{2\pi T}(u + re_T)\|_{L^\alpha_{2\pi T}} \le \overline{c}_\alpha\|u + re_T\|_{L^\alpha_{2\pi T}}.$$

Thus we have

$$I_T(u + re_T) \le \frac{r^2}{2}\|e_T\|^2_{E_{2\pi T}} - \frac{1}{2}\|u\|^2_{E_{2\pi T}} - k_1(\overline{c}_\alpha)^{-\alpha}r^\alpha\|e_T\|^\alpha_{L^\alpha_{2\pi T}}.$$

We can easily deduce the desired result from Lemma 1.4. ∎

Now we can apply Proposition 1.3 to $I_T(z)$ and we get

Proposition 1.7. *For any $T \in \mathbf{N}$, there is a nontrivial critical point $z_T(t) \in E_{2\pi T}$ (i.e., a solution of* (HS:T) *) and its critical value $b_T = I_T(z_T)$ is characterized as*

$$b_T = \inf_{h\in\Gamma_T}\sup_{z\in N_{T,R}} I_T(h(1,z)) \ge m > 0, \tag{1.16}$$

where Γ_T is defined in a similar way to Γ. Moreover there are constants m, $M > 0$ independent of $T \in \mathbf{N}$ such that

$$m \le b_T \equiv I_T(z_T) \le M \quad \text{for all } T \in \mathbf{N}. \tag{1.17}$$

Proof. We need only to prove the right hand side inequality of (1.17). Since $id \in \Gamma_T$, we have from (1.16) that

$$b_T \le \sup_{z\in N_{T,R}} I_T(z).$$

By (1.15), we obtain (1.17). ∎

Remark 1.1. A regularity argument shows $z_T(t) \in C^1(\mathbf{R}, \mathbf{R}^{2N})$. (c.f. Chapter 6 of [**8**].)

Thus by Proposition 1.7 we obtain $2\pi T$-periodic solutions $z_T(t)$ of (HS) with properties (i), (ii) of Theorem 0.1. In the following section, we verify the compactness property (iii) of Theorem 0.1 for $(z_T(t))_{T\in\mathbf{N}}$.

2. Uniform estimates and limit process for $z_T(t)$

In this section, we consider the behavior of $z_T(t)$ as $T \to \infty$. Firstly we establish some uniform estimates for $z_T(t)$ and secondly we pass to the limit as $T \to \infty$ and complete the proof of Theorem 0.3.

In what follows, we denote by C, C_0, C_1, $\cdots$ various constants which are independent of $T \in \mathbf{N}$.

2.1. Uniform estimates for $z_T(t)$

Let $z_T(t)$ be a solution of (HS) obtained in Proposition 1.7; especially $z_T(t)$ satisfies

$$I_T'(z_T) = 0, \tag{2.1}$$

$$I_T(z_T) \in [m, M] \quad \text{for all } T \in \mathbf{N}. \tag{2.2}$$

The following lemma provides uniform estimates of $z_T(t)$ from above.

Lemma 2.1. *There is a constant $C > 0$ independent of $T \in \mathbf{N}$ such that*

$$\|z_T\|_{E_{2\pi T}} \leq C \quad \textit{for all } T \in \mathbf{N}. \tag{2.3}$$

Proof. We write $z_T = z_T^+ + z_T^- \in E_{2\pi T}^+ \oplus E_{2\pi T}^-$. We have by (2.1), (2.2) and (W2)

$$\begin{aligned}
M &\geq I_T(z_T) - \frac{1}{2} I_T'(z_T) z_T \\
&= \int_0^{2\pi T} ((\frac{1}{2} W_z(t, z_T), z_T) - W(t, z_T)) dt \\
&\geq (\frac{\mu}{2} - 1) \int_0^{2\pi T} W(t, z_T) dt.
\end{aligned}$$

Thus we get

$$\int_0^{2\pi T} W(t, z_T) dt \leq C_1,$$

i.e.,

$$\|z_T\|_{L^\alpha_{2\pi T}} \leq C_2. \tag{2.4}$$

On the other hand, we have

$$0 = I_T'(z_T)(z_T^+ - z_T^-) = \|z_T\|^2_{E_{2\pi T}} - \int_0^{2\pi T} (W_z(t, z_T), z_T^+ - z_T^-) dt,$$

i.e.,

$$\|z_T\|^2_{E_{2\pi T}} = \int_0^{2\pi T} (W_z(t, z_T), z_T^+ - z_T^-) dt.$$

Remarking $\frac{\alpha}{\beta-1} \in (1,2)$, we have from (1.9)

$$\begin{aligned}
\|z_T\|^2_{E_{2\pi T}} &\le \|W_z(t,z_T)\|_{L^{\alpha/(\beta-1)}_{2\pi T}} \|z_T^+ - z_T^-\|_{L^{\alpha/(\alpha-\beta+1)}_{2\pi T}} \\
&\le C_3 \|W_z(t,z_T)\|_{L^{\alpha/(\beta-1)}_{2\pi T}} \|z_T^+ - z_T^-\|_{E_{2\pi T}} \\
&= C_3 \|W_z(t,z_T)\|_{L^{\alpha/(\beta-1)}_{2\pi T}} \|z_T\|_{E_{2\pi T}},
\end{aligned}$$

i.e.,

$$\|z_T\|_{E_{2\pi T}} \le C_3 \|W_z(t,z_T)\|_{L^{\alpha/(\beta-1)}_{2\pi T}}.$$

By (W4"),

$$\|z_T\|_{E_{2\pi T}} \le C_4 \|z_T\|^{\beta-1}_{L^{\alpha}_{2\pi T}}.$$

From (2.4), we get

$$\|z_T\|_{E_{2\pi T}} \le C_4 c_2^{\beta-1}.$$

Therefore we get the desired result. ∎

Corollary 2.2. *There is a constant $\overline{M} > 0$ independent of $T \in \mathbf{N}$ such that*

$$\|z_T\|_{C^1(\mathbf{R},\mathbf{R}^{2N})} \le \overline{M} \quad \text{for all } T \in \mathbf{N}. \tag{2.5}$$

Proof. By (iii) of Proposition 1.1, we have from $\Phi_{2\pi T} z_T(t) = -JW_z(t, z_T(t))$ that

$$\begin{aligned}
\|z_T\|_{L^\infty} &\le c\|\Phi_{2\pi T} z_T\|_{L^2_{2\pi T}} \\
&= c\|W_z(t,z_T)\|_{L^2_{2\pi T}} \\
&\le ck_2 \|z_T\|^{\beta-1}_{L^{2(\beta-1)}_{2\pi T}} \\
&\le ck_2 c^{\beta-1}_{2(\beta-1)} \|z_T\|^{\beta-1}_{E_{2\pi T}}.
\end{aligned}$$

Thus we can get $\|z_T\|_{L^\infty} \le C$ from (2.3). Since $z_T(t)$ satisfies (HS), we get (2.5). ∎
Next we obtain uniform estimate of $\|z_T\|_{L^\infty}$ from below.

Lemma 2.3. *There is a constant $\delta > 0$, which is independent of $T \in \mathbf{N}$ such that*

$$\|z_T\|_{L^\infty} \ge \delta \quad \text{for all } T \in \mathbf{N}. \tag{2.6}$$

Proof. By the assumption (W4"), for any $\epsilon > 0$ we can find a $\delta_\epsilon > 0$ such that

$$| W_z(t,z) | \le \epsilon | z | \quad \text{for} \quad | z | \le \delta_\epsilon. \tag{2.7}$$

Suppose that $\|z_T\|_{L^\infty_{2\pi T}} \le \delta_\epsilon$. Then, using (2.7), we have as in the proof of Lemma 2.1

$$\begin{aligned}\|z_T\|^2_{E_{2\pi T}} &= \int_0^{2\pi T} (W_z(t, z_T), z_T^+ - z_T^-)dt \\ &\le \epsilon \int_0^{2\pi T} | z || z_T^+ - z_T^- | \, dt \\ &\le \epsilon \|z_T\|^2_{L^2_{2\pi T}} \\ &\le \epsilon c_2^2 \|z_T\|^2_{E_{2\pi T}}\end{aligned}$$

Choosing $\epsilon \in (0, 1/c_2^2)$, we have $z_T = 0$. But this contradicts with $I_T(z_T) \ge m > 0$. Therefore we have (2.6). ∎

2.2. Limit process for $z_T(t)$ — Proof of Theorem 0.3

We can find a sequence $(\ell_T)_{T\in\mathbf{N}}$ of integers such that

$$\max_{t\in[0,2\pi]} | z_T(t + 2\pi\ell_T) | = \max_{t\in\mathbf{R}} | z_T(t) | \in [\delta, \overline{M}]. \tag{2.8}$$

We remark that $\widetilde{z}_T(t) \equiv z_T(t+2\pi\ell_T)$ is a solution of (HS) satisfying (i), (ii) of Theorem 0.1 and $I_T(\widetilde{z}_T) = I_T(z_T)$. In what follows, we show that $(\widetilde{z}_T(t))_{T\in\mathbf{N}}$ possesses the compactness property (iii) of Theorem 0.1.

By Corollary 2.2, we can extract a subsequence from any given sequence of integers $T_n \to \infty$ — we still denote it by T_n — such that

$$\widetilde{z}_{T_n} \equiv z_{T_n}(t + 2\pi\ell_{T_n}) \to z_\infty(t) \quad \text{in } C^1_{loc}(\mathbf{R}, \mathbf{R}^{2N}), \tag{2.9}$$

where $z_\infty(t) \in C^1(\mathbf{R}, \mathbf{R}^{2N})$ is a solution of (HS). The following Lemma 2.4 completes the proof of Theorem 0.3.

Lemma 2.4. *$z_\infty(t)$ satisfies the following*

(i) $z_\infty(t) \not\equiv 0$.

(ii) $z_\infty(t) \in L^p(\mathbf{R}, \mathbf{R}^{2N})$ *for all* $p \in [2, \infty]$.

(iii) $| z_\infty(t) |, | \dot{z}_\infty(t) | \to 0$ *as* $| t | \to \infty$.

Proof. (i) By (2.8) and (2.9), we have

$$\max_{t\in[0,2\pi]} | z_\infty(t) | = \sup_{t\in\mathbf{R}} | z_\infty(t) | \in [\delta, \overline{M}]. \tag{2.10}$$

Therefore we have (i).

(ii) For any $R > 0$ and $p \in [2, \infty)$, we have from (1.9)

$$\begin{aligned}\int_{-R}^{R} | z_\infty(t) |^p \, dt &= \lim_{n\to\infty} \int_{-R}^{R} | z_{T_n}(t + 2\pi\ell_{T_n}) |^p \, dt \\ &\le \limsup_{T\to\infty} \|z_T\|^p_{L^p_{2\pi T}} \\ &\le C \limsup_{T\to\infty} \|z_T\|^p_{E_{2\pi T}} \\ &\le C_p.\end{aligned}$$

Since $C_p > 0$ is independent of $R > 0$, we get (ii) for $p \in [2, \infty)$. For $p = \infty$, (ii) follows from (2.10).

(iii) Let F_s (resp. F_u) be a stable (resp. unstable) subspace of the flow defined by $\dot{z} = JAz$, i.e., $\mathbf{R}^{2N} = F_s \oplus F_u$, F_s and F_u are invariant under JA and

$$\begin{aligned}| e^{t(JA)}x | &\le Ce^{-at} \quad \text{for } t \ge 0 \text{ and } x \in F_s, \\ | e^{-t(JA)}y | &\le Ce^{-at} \quad \text{for } t \ge 0 \text{ and } y \in F_u,\end{aligned} \tag{2.11}$$

where $C > 0$ and $a > 0$ are constants independent of x and y. Since $z_\infty(t)$ is a solution of (HS) on $\mathbf{R}$, we have

$$\begin{aligned}z_\infty(t) = e^{t(JA)}z_0 &+ \int_{-\infty}^{t} e^{(t-\tau)JA}\widetilde{P}_s JW_z(\tau, z_\infty(\tau))d\tau \\ &- \int_{t}^{\infty} e^{(t-\tau)JA}\widetilde{P}_u JW_z(\tau, z_\infty(\tau))d\tau\end{aligned} \tag{2.12}$$

for some $z_0 \in \mathbf{R}^{2N}$. Here, $\widetilde{P}_s : \mathbf{R}^{2N} \to F_s$ and $\widetilde{P}_u : \mathbf{R}^{2N} \to F_u$ are projections. By (ii) and (W4"), we have $z_\infty(t) \in L^2(\mathbf{R}, \mathbf{R}^{2N})$ and $W_z(t, z_\infty(t)) \in L^2(\mathbf{R}, \mathbf{R}^{2N})$. Hence we see by (2.11) that

$$\begin{aligned}&\int_{-\infty}^{t} e^{(t-\tau)JA}\widetilde{P}_s JW_z(\tau, z_\infty(\tau))d\tau \in L^2(\mathbf{R}, \mathbf{R}^{2N}), \\ &\int_{t}^{\infty} e^{(t-\tau)JA}\widetilde{P}_u JW_z(\tau, z_\infty(\tau))d\tau \in L^2(\mathbf{R}, \mathbf{R}^{2N}).\end{aligned}$$

On the other hand, we have $e^{t(JA)}z \notin L^2(\mathbf{R}, \mathbf{R}^{2N})$ for $z \ne 0$. Thus $z_0 = 0$ follows from (2.12). Therefore we can easily deduce from (2.12) that $z_\infty(t) \to 0$ as $| t | \to \infty$. Since $z_\infty(t)$ satisfies (HS) we have

$$\dot{z}_\infty(t) = JH_z(t, z_\infty(t)) \to 0 \quad \text{as } | t | \to \infty.$$

Therefore the proof is completed. ∎

3. Proof of Proposition 1.1

This section is devoted to prove Proposition 1.1. Using Fourier series, we have the following representation of $\Phi_{2\pi T}$

$$(\Phi_{2\pi T} z)(t) = \sum_{j\in\mathbf{Z}} (-\frac{ij}{T} J - A) a_j e^{ijt/T} \tag{3.1}$$

where

$$z(t) = \sum_{j\in\mathbf{Z}} a_j e^{ijt/T} \quad (a_j \in \mathbf{C}^{2N} \text{ with } a_{-j} = \overline{a_j}). \tag{3.2}$$

We also have

$$\|z(t)\|^2_{L^2_{2\pi T}} = 2\pi T \sum_{j\in\mathbf{Z}} | a_j |^2 . \tag{3.3}$$

We remark that $\mathrm{span}\{a e^{ijt/T} + \overline{a} e^{ijt/T}; a \in \mathbf{C}^{2N}\}$ is invariant under $\Phi_{2\pi T}$ for all $j \in \mathbf{N}$ and $E_{2\pi T} = D(| \Phi_{2\pi T} |^{1/2})$ can be written

$$E_{2\pi T} = \{z = \sum_{j\in\mathbf{Z}} a_j e^{ijt/T};\ \|z\|^2_{E_{2\pi T}} \equiv 2\pi T \sum_{j\in\mathbf{Z}} (| \frac{ij}{T} J + A | a_j, a_j) < \infty\}. \tag{3.4}$$

where $(x, y) = \sum_{k=1}^{2N} x_k \overline{y_k}$ for $x = (x_1, \cdots, x_{2N})$, $y = (y_1, \cdots, y_{2N}) \in \mathbf{C}^{2N}$. Note that $-i\theta J - A$ $(\theta \in \mathbf{R})$ is a $2N \times 2N$ Hermitian matrix and we can define $| i\theta J + A |$: $\mathbf{C}^{2N} \to \mathbf{C}^{2N}$.

By the assumption (A), we have

$$0 \notin \sigma(-i\theta J - A) \quad \text{for all } \theta \in \mathbf{R}. \tag{3.5}$$

We can see that $-i\theta J - A$ has N positive eigenvalues and N negative eigenvalues (counting multiplicities). In fact, eigenvalues are solutions of

$$\det(\lambda + (i\theta J + A)) = 0.$$

By (3.5), we can see the number of positive (or negative) eigenvalues is independent of $\theta \in \mathbf{R}$. Dividing by $\theta > 0$, it equals to the number of positive (or negative) solutions of

$$\det(\lambda + (iJ + \frac{1}{\theta} A)) = 0.$$

Passing to the limit as $\theta \to \infty$, we see it equals to N. We denote the eigenvalues of $-i\theta J - A$ by $\lambda_N^-(\theta) \le \cdots \le \lambda_1^-(\theta) < 0 < \lambda_1^+(\theta) \le \cdots \le \lambda_N^+(\theta)$ and the corresponding eigenvectors by $\xi_N^-(\theta), \cdots, \xi_1^-(\theta), \xi_1^+(\theta), \cdots, \xi_N^+(\theta)$. We remark

$$\lambda_k^\pm(-\theta) = \lambda_k^\pm(\theta), \tag{3.6}$$

and

$$\xi_k^\pm(-\theta) = \overline{\xi_k^\pm(\theta)} \tag{3.7}$$

for all $\theta \in \mathbf{R}$ and $k = 1, \cdots, N$.

Lemma 3.1. *Under the assumption (A), there are constants c, $c' > 0$ independent of $\theta \in \mathbf{R}$ such that*

$$c(1+ \mid \theta \mid) \leq \mid \lambda_k^{\pm}(\theta) \mid \leq c'(1+ \mid \theta \mid) \tag{3.8}$$

for all $\theta \in \mathbf{R}$ and $k = 1, \cdots, N$.

Remark 3.1. (1.4) follows from (3.8).

Proof. Since $\lambda_k^{\pm}(\theta)$ is a solution of

$$\det(\frac{\lambda}{\theta} + (iJ + \frac{1}{\theta}A)) = 0,$$

it is clear that

$$\mid \frac{\lambda_k^{\pm}(\theta)}{\theta} \mid \to 1 \quad \text{as } \mid \theta \mid \to \infty. \tag{3.9}$$

On the other hand, by (3.5) we have

$$\begin{aligned} 0 &< \inf\{\mid \lambda_k^{\pm}(\theta) \mid; \mid \theta \mid \leq L,\, 1 \leq k \leq N\} \\ &\leq \sup\{\mid \lambda_k^{\pm}(\theta) \mid; \mid \theta \mid \leq L,\, 1 \leq k \leq N\} < \infty \end{aligned} \tag{3.10}$$

for any $L > 0$. Combining (3.9) and (3.10), we get (3.8). ∎

Now we can prove (i), (ii), (iii) of Proposition 1.1.

Proof of (i) of Proposition 1.1. By (3.8), we have

$$c\sum_{j\in\mathbf{Z}}(1 + \frac{\mid j \mid}{T}) \mid a_j \mid^2 \leq \sum_{j\in\mathbf{Z}}(\mid \frac{ij}{T}J + A \mid a_j, a_j) \leq c' \sum_{j\in\mathbf{Z}}(1 + \frac{\mid j \mid}{T}) \mid a_j \mid^2 .$$

Thus by the definition of $\|z\|_{H^{1/2}_{2\pi T}}$ and (3.4), we get (1.8) and $E_{2\pi T} = H^{1/2}_{2\pi T}$. ∎

Proof of (ii) of Proposition 1.1. It is suffices to prove

$$\|z\|_{L^p_{2\pi T}} \leq c_p \|z\|_{H^{1/2}_{2\pi T}} \quad \text{for } z \in H^{1/2}_{2\pi T}. \tag{3.11}$$

For $z(t)$ of form (3.2), we have from Hausdorff-Young's inequality and Hölder's inequality,

$$\begin{aligned} \|z\|_{L^p_{2\pi T}} &\leq (2\pi T)^{1/p} (\sum_{j\in\mathbf{Z}} \mid a_j \mid^q)^{1/q} \\ &\leq (2\pi T)^{1/p} (\sum_{j\in\mathbf{Z}} (1 + \frac{\mid j \mid}{T})^{-q/(2-q)})^{(2-q)/2q} (\sum_{j\in\mathbf{Z}} (1 + \frac{\mid j \mid}{T}) \mid a_j \mid^2)^{1/2} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Since

$$\sum_{j\in\mathbf{Z}}(1 + \frac{\mid j \mid}{T})^{-q/(2-q)} \leq 1 + \int_{\mathbf{R}} (1 + \frac{\mid s \mid}{T})^{-q/(2-q)} ds = 1 + c_q T,$$

we get (3.11). ∎

Proof of (iii) of Proposition 1.1. In a similar way to the proof of (i) of Proposition 1.1, we can see for some constants c, $c' > 0$,

$$c\|z\|_{H^1_{2\pi T}} \leq \|\Phi_{2\pi T} z\|_{L^2_{2\pi T}} \leq c'\|z\|_{H^1_{2\pi T}} \quad \text{for all } z \in D(\Phi_{2\pi T}). \tag{3.12}$$

Here

$$\begin{aligned} \|z\|^2_{H^1_{2\pi T}} &= \int_0^{2\pi T} (|\, u\,|^2 + |\, \dot{u}\,|^2) dt \\ &= 2\pi T \sum_{j\in\mathbf{Z}} (1 + \frac{|\, j\,|}{T})^2 \,|\, a_j \,|^2 . \end{aligned}$$

As in the proof of (ii) of Proposition 1.1, we get

$$\|z\|_{L^\infty} \leq c''\|z\|_{H^1_{2\pi T}} \tag{3.13}$$

where $c'' > 0$ is independent of $T \in \mathbf{N}$ and $z \in D(\Phi_{2\pi T})$. Combining (3.12) and (3.13), we get (iii) of Proposition 1.1. ∎

Next we give a proof to (iv) of Proposition 1.1. We write $P^{\pm}_{2\pi T} : E_{2\pi T} \to E^{\pm}_{2\pi T}$ by means of Fourier series; let $Q^{\pm}_{\theta}$ be a matrix associated to the projection $\mathbf{C}^{2N} \to \mathrm{span}\{\xi^{\pm}_k(\theta);\, 1 \leq k \leq N\}$. Then we can see form (3.1)

$$(P^{\pm}_{2\pi T} z)(t) = \sum_{j\in\mathbf{Z}} (Q^{\pm}_{j/T} a_j) e^{ijt/T} \tag{3.14}$$

for $z(t)$ of form (3.2). By (3.6) and (3.7), we remark

$$Q^{\pm}_{-j/T} a_{-j} = \overline{Q^{\pm}_{j/T} a_j} \quad \text{for all } j \in \mathbf{Z}.$$

We can easily see that from (3.14) that

$$\begin{aligned} \|P^{\pm}_{2\pi T} z\|_{L^2_{2\pi T}} &\leq \|z\|_{L^2_{2\pi T}}, \\ \|P^{\pm}_{2\pi T} z\|_{E_{2\pi T}} &\leq \|z\|_{E_{2\pi T}} \end{aligned} \tag{3.15}$$

for all $z \in E_{2\pi T}$.

To prove the continuity of $P^{\pm}_{2\pi T} : L^p_{2\pi T} \to L^p_{2\pi T}$, we introduce the following operator

$$\widehat{P_{2\pi T}}^{\pm} : L^2_{2\pi} \to L^2_{2\pi}$$

defined by

$$(\widehat{P_{2\pi T}}^{\pm} z)(t) = \sum_{j\in\mathbf{Z}} (Q^{\pm}_{j/T} a_j) e^{ijt} \tag{3.16}$$

for

$$z(t) = \sum_{j \in \mathbf{Z}} a_j e^{ijt} \quad (a_j \in \mathbf{C}^{2N} \text{ with } a_{-j} = \overline{a_j}).$$

Since

$$\begin{aligned} &\sup\{\|P^{\pm}_{2\pi T} z\|_{L^p_{2\pi T}};\ z \in L^p_{2\pi T},\ \|z\|_{L^p_{2\pi T}} \le 1\} \\ =&\sup\{\|\widehat{P_{2\pi T}}^{\pm} z\|_{L^p_{2\pi}};\ z \in L^p_{2\pi},\ \|z\|_{L^p_{2\pi}} \le 1\}, \end{aligned} \tag{3.17}$$

we estimate the right hand side instead of the left hand side. We relay on the following Stečkin's theorem (Theorem 3.5 of [**3**]).

Proposition 3.2. *Let $(\phi(j))_{j \in \mathbf{Z}}$ be a function of bounded variation on $\mathbf{Z}$. Then for each $p \in (1, \infty)$ the operator*

$$(T_\phi z)(t) = \sum_{j \in \mathbf{Z}} \phi(j) a_j e^{ijt} \quad \text{for } z(t) = \sum_{j \in \mathbf{Z}} a_j e^{ijt}$$

is continuous as $L^p_{2\pi} \to L^p_{2\pi}$. Moreover there is a constant $C_p > 0$ independent of ϕ such that

$$\sup_{\|z\|_{L^p_{2\pi}} = 1} \|T_\phi z\|_{L^p_{2\pi}} \le C_p \max\{|\ \phi(0)\ |,\ \mathrm{Var}\,\phi\} \tag{3.18}$$

∎

Proof of (iv) of Proposition 1.1. We apply Theorem 3.2 to (3.16). By (3.17) and (3.18), we need only to prove the existence of $C_0 > 0$ such that

$$\mathrm{Var}(Q^{\pm}_{j/T}) \le C_0 \quad \text{for all } T \in \mathbf{N}. \tag{3.19}$$

We have

$$\begin{aligned} \mathrm{Var}(Q^{\pm}_{j/T}) &\equiv \sum_{j \in \mathbf{Z}} |\ Q^{\pm}_{(j+1)/T} - Q^{\pm}_{j/T}\ | \\ &\le \int_{-\infty}^{\infty} |\ \frac{dQ^{\pm}_\theta}{d\theta}\ |\ d\theta. \end{aligned} \tag{3.20}$$

In what follows, we see the right hand side is finite (clearly it is independent of $T \in \mathbf{N}$). We deal with only "+" case. The case "−" is treated similarly. First we prove $\int_1^\infty |\ \frac{dQ^+_\theta}{d\theta}\ |\ d\theta < \infty$.

Since Q^+_θ is a projection operator corresponding to $-i\theta J - A$, it is also corresponding to $-iJ - \frac{1}{\theta}A$. By Lemma 3.1, we can find constants a, $b > 0$ independent of $\theta \in [1, \infty)$ such that

$$a \le \frac{\lambda^+_k(\theta)}{\theta} \le b \quad \text{for all } \theta \in [1, \infty) \text{ and } k = 1, \cdots, N.$$

Since $\lambda_k^+(\theta)/\theta$ are eigenvalues of $-iJ - \frac{1}{\theta}A$, we have

$$Q_\theta^+ = \frac{1}{2\pi i}\int_\gamma (\zeta + iJ + \frac{1}{\theta}A)^{-1} d\zeta.$$

Here, γ is a cycle in the right half plane $\{z \in \mathbf{C};\ \mathrm{Re}\, z > 0\}$ which surrounds the interval $[a, b]$. Thus

$$\frac{dQ_\theta^+}{d\theta} = \frac{1}{2\pi i}\int_\gamma \theta^{-2}(\zeta + iJ + \frac{1}{\theta}A)^{-1}A(\zeta + iJ + \frac{1}{\theta}A)^{-1} d\zeta.$$

Hence we have

$$\left| \frac{dQ_\theta^+}{d\theta} \right| \leq \frac{1}{2\pi}\int_\gamma \theta^{-2} \mid A \mid\mid (\zeta + iJ + \frac{1}{\theta}A)^{-1} \mid^2 \mid d\zeta \mid \leq C\theta^{-2},$$

where $C > 0$ is independent of $\theta \geq 1$. Therefore we have

$$\int_1^\infty \left| \frac{dQ_\theta^+}{d\theta} \right| d\theta < \infty. \tag{3.21}$$

Using representation

$$Q_\theta^+ = \frac{1}{2\pi i}\int_{\gamma'} (\zeta + i\theta J + A)^{-1} d\zeta,$$

where γ' is a cycle in $\{z \in \mathbf{C};\ \mathrm{Re}\, z > 0\}$ surrounding the set $\{\xi_k^+(\theta);\ k = 1, \cdots, N, \mid \theta \mid \leq 1\}$, we obtain

$$\int_{-1}^1 \left| \frac{dQ_\theta^+}{d\theta} \right| d\theta < \infty. \tag{3.22}$$

Similarly to (3.21), we obtain

$$\int_{-\infty}^{-1} \left| \frac{dQ_\theta^+}{d\theta} \right| d\theta < \infty. \tag{3.23}$$

Combining (3.21)–(3.23), we obtain

$$\int_{-\infty}^\infty \left| \frac{dQ_\theta^+}{d\theta} \right| d\theta < \infty.$$

Thus we obtain (3.19). ∎

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